Generalized Berry Conjecture and mode correlations in chaotic plates

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(Dated: 2nd February 2008)

We consider a modification of the Berry Conjecture for eigenmode statistics in wave-bearing systems. The eigenmode correlator is conjectured to be proportional to the imaginary part of the Green's function. The generalization is applicable not only to scalar waves in the interior of homogeneous isotropic systems where the correlator is a Bessel function, but to arbitrary points of heterogeneous systems as well. In view of recent experimental measurements, expressions for the intensity correlator in chaotic plates are derived.

In 1977 Berry conjectured that the higher eigenmodes of a ray-chaotic Hamiltonian, in particular, a billiard, should be statistically indistinguishable from a superposition of plane standing waves of all directions, with uncorrelated amplitudes and phases [1]; the idea is also found in [2]. One immediate consequence is that the modes are Gaussian random functions, with correlations given by

$$\left\langle u^{(n)}\left(\mathbf{x}\right) u^{(m)}\left(\mathbf{x}+\mathbf{r}\right)\right\rangle = A^2 \delta_{nm} J_0\left(kr\right),$$

where brackets $\langle\cdot\rangle$ represent spatial average over position $\mathbf{x}$, A is an unimportant normalization factor, $r = |\mathbf{r}|$ is a separation distance, and k is the wavenumber that appears in the governing Helmholtz equation, $\left(\nabla^2 + k^2\right) u^{(n)}\left(\mathbf{x}\right) = 0$. Another immediate consequence is that the intensity correlator is

$$\left\langle u^{(n)}\left(\mathbf{x}\right)^2 u^{(n)}\left(\mathbf{x}+\mathbf{r}\right)^2\right\rangle = A^4\left[1 + 2J_0^2\left(kr\right)\right]. \qquad (1)$$

Berry established the conjecture for an asymptotic regime in which wavelengths are much less than system size. It is only approximate at finite wavelength, i.e, in practice. The conjecture has been shown to be incorrect at finite wavelength, inasmuch as many modes show evidence of scarring [3], existence of which may be understood in the light of the "quantum equidistribution theorem" put forth by Shnirelman [4]. The conjecture is manifestly incorrect if attention is restricted to points near a boundary where, locally, plane waves are correlated with their reflections. Nevertheless, numerical and experimental evidence shows that it is widely satisfied [5], and references in [6].

Recent measurements on elastic waves in plates have underlined the inadequacy of the conjecture, as stated, for systems more complicated than the scalar billiard [7]. Three dimensional microwave billiards (or merely thick quasi 2D billiards) for which the electric field satisfies a vector wave equation, and elastic wave systems in general, require a statement about the correlations of the vector-valued eigenmodes [8]. Even in scalar wave systems, if they are inhomogeneous, or if interest includes points near boundaries, the conjecture needs modification. It is not sufficient to extend the conjecture by expressing the modes as uncorrelated superpositions of plane waves of all wave types, as their relative amplitudes remain unspecified. In those recent measurements on elastic waves in plates, the intensity correlator did not satisfy (1). This note is intended to provide the appropriate generalization of Berry's conjecture needed for experiments in wave-bearing systems more complex than scalar billiards.

We begin with an identity, written in the form it takes for a tensor Green's function appropriate for a vector wave equation,

$$\mathbf{G}\left(\mathbf{x}_1, \mathbf{x}_2, \omega\right) = \sum_n \frac{\mathbf{u}^{(n)}\left(\mathbf{x}_1\right) \otimes \mathbf{u}^{(n)}\left(\mathbf{x}_2\right)}{\omega_n^2 - \left(\omega + \imath\varepsilon\right)^2}, \qquad (2)$$

as a modal sum over the normalized real modes with eigenfrequencies ω_n. The imaginary part of this Green's dyadic is [9]

$$\Im\mathbf{G}\left(\mathbf{x}_1, \mathbf{x}_2, \omega\right) = \frac{\pi}{2\omega} \sum_n \mathbf{u}^{(n)}\left(\mathbf{x}_1\right) \otimes \mathbf{u}^{(n)}\left(\mathbf{x}_2\right) \delta\left(\omega - \omega_n\right).$$

This may be averaged, either over a short range in frequency, or over an ensemble of systems that differ from the system of interest only at positions far from the closely spaced points $\mathbf{x}_1$ and $\mathbf{x}_2$. In either case $\mathbf{G}$ is largely unaffected. The right side becomes the corresponding modal correlator. It is seen to be in general not J_0, but rather $\Im\mathbf{G}$. It is not a spatial average, as called for by the Berry conjecture, but rather a frequency or ensemble average.

Thus we are led to a generalized Berry conjecture. Based upon the exact identity for ensemble or frequency averages, we conjecture that it is also true for spatial averages at a fixed mode. This reduces to the Berry conjecture for the simple case of a scalar wave. The conjecture about the correlator is, as demonstrated above, manifestly correct if what one means by the averaging is a frequency or ensemble average. The potentially more problematic aspects lie in the supposition that this correlator may be found within spatial averages on a single mode of a single sample from the ensemble, or for that matter that the statistics are Gaussian.

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When this generalized Berry conjecture is applied to the modes of an infinite isotropic homogeneous 3D elastic body, the modal correlator is given by the Green's function, $\mathbf{G}^\infty$, satisfying Navier's equations [10]:

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{G}^\infty)+\mu\nabla^2\mathbf{G}^\infty+\rho\omega^2\mathbf{G}^\infty=-\mathbf{I}\delta^3(\mathbf{r}),$$

with $\mathbf{I}$ being the identity tensor, λ and μ being elastic Lamé constants, and ρ representing material density. The spatial Fourier transform of the solution is obtained readily:

$$\rho\mathbf{G}^\infty(\mathbf{q},\omega)=\frac{\mathbf{q}\otimes\mathbf{q}/\left|\mathbf{q}\right|^2}{\left|\mathbf{q}\right|^2c_l^2-(\omega+\imath\varepsilon)^2}+\frac{\mathbf{I}-\mathbf{q}\otimes\mathbf{q}/\left|\mathbf{q}\right|^2}{\left|\mathbf{q}\right|^2c_t^2-(\omega+\imath\varepsilon)^2}.$$

The first term is the longitudinal part with wavespeed c_l, the second is the transverse part with wavespeed c_t. On taking the imaginary part of the 3D inverse Fourier transform, one finds

$$\Im\mathbf{G}^\infty\propto\int\cos(\mathbf{q}\cdot\mathbf{r})\Big[c_l^{-3}\delta(|\mathbf{q}|-\omega/c_l)\,\mathbf{q}\otimes\mathbf{q}/\left|\mathbf{q}\right|^2$$
$$+c_t^{-3}\delta(|\mathbf{q}|-\omega/c_t)\left(\mathbf{I}-\mathbf{q}\otimes\mathbf{q}/\left|\mathbf{q}\right|^2\right)\Big]d^2\Omega_{\mathbf{q}}dq.$$

It is seen that $\Im\mathbf{G}^\infty$ is a superposition of plane waves of the two types and of all directions of propagation, with relative strengths given by the inverse cubes of the wave speeds, i.e. by equipartition [11].

If the frequency averaging is over a sufficiently broad band, then, even in a finite system, the correlator reduces to that in the unbounded medium, $\Im\mathbf{G}^\infty$. This is readily established, as in [12], by recognizing that short time responses are independent of distant parts of a structure, and that sufficiently short time responses are equivalent to frequency averaging over bands of sufficient width. The theorem is readily generalized to the vicinity of a boundary, or a scatterer. In these cases it is not difficult to show that the diffuse field-field correlator may be also constructed by superposing an equipartitioned set of uncorrelated incident plane or incoming or standing waves together with their coherent reflections and scatterings. For the special case of elastic waves near a free surface, this was explored in [13].

The Green's function is more complicated in a plate, in particular if its thickness is comparable to a wavelength. In the work reported by Schaadt *et al.* [7], an intensity correlator was constructed by averages over space and over a small number of modes. Due to the good preservation of up/down symmetry, all modes were either of a purely flexural (odd up/down parity) character, or a mixture of extensional and shear (even up/down parity). Thus their correlator should be the imaginary parts of the partial Green's functions. Except for the effects of nonzero thickness to wavelength ratio, these modes have displacements that are purely out-of-plane or purely in-plane respectively.

Modes which are antisymmetric on reflection about the mid-plane, sometimes called flexural, are un-coupled to the others; at frequencies below the first cutoff ($\omega=\pi c_t/2h$, where h is the half-width of the plate) there is a single wave number k_f governing such waves. Their intensity correlator must therefore be formed from a single Bessel function $J_0(k_f r)$. This was in fact observed [7]; Schaadt *et al.* reported a good fit to $1+2J_0^2(k_f r)$. These modes in fact have vector valued fields, so the correlator is technically a tensor. The measured correlator is a contraction of that tensor with the (unknown) polarization vector $\hat{\mathbf{p}}$ of the detector. Given a field (an eigenmode) $\psi(\mathbf{x})$ they constructed an 'intensity correlator'

$$I(r)=\overline{\left\langle[\hat{\mathbf{p}}\cdot\psi(\mathbf{x})]^2[\hat{\mathbf{p}}\cdot\psi(\mathbf{x}+\mathbf{r})]^2\right\rangle\Big/\left\langle[\hat{\mathbf{p}}\cdot\psi(\mathbf{x})]^2\right\rangle^2},$$

where the overbar indicates additional average over the direction of the vector $\mathbf{r}$. We presume that the detector polarization $\hat{\mathbf{p}}$ is held fixed during this averaging. Inasmuch as $\mathbf{u}$ is a Gaussian process, the above fourth order statistic reduces to the sum of three products of two second order statistics. In terms of the generalized Berry conjecture the correlator is rewritten as

$$I(\mathbf{r})=1+2\overline{\langle\hat{\mathbf{p}}\cdot\Im\mathbf{G}(\mathbf{x},\mathbf{x}+\mathbf{r})\cdot\hat{\mathbf{p}}\rangle^2}\Big/\langle\hat{\mathbf{p}}\cdot\Im\mathbf{G}(\mathbf{x},\mathbf{x})\cdot\hat{\mathbf{p}}\rangle^2. \tag{3}$$

For the surface of a plate in flexure, even far from the edges, such that $\mathbf{G}\approx\mathbf{G}^\infty$, the Green's function is not as simple as might have been hoped,

$$\begin{aligned}\Im\mathbf{G}^\infty(\mathbf{r})\propto&\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_1\nu^2\left[J_0(k_f r)-J_2(k_f r)\right]/2\\&+\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_2\nu^2\left[J_0(k_f r)+J_2(k_f r)\right]/2\\&+(\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_3-\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_1)\,\nu J_1(k_f r)\\&+\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_3J_0(k_f r).\end{aligned}$$

Unit vectors $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ lie in the mid-plane of the plate, with $\hat{\mathbf{x}}_1$ taken in the direction of $\mathbf{r}$, and $\hat{\mathbf{x}}_3$ is normal to the plane and pointing towards the surface at hand. Factor ν represents a degree of in-plane surface motion associated with such waves, and vanishes at long wavelength. The correlator I is not only dependent upon separation distance r, but also upon the angle between polarization of the detector $\hat{\mathbf{p}}$ and the separation vector direction $\mathbf{r}/r$.

By averaging over the direction of vector $\mathbf{r}$, for a single flexural mode, one finds (see Appendix)

$$I(r)=1+2\frac{J_0^2(k_f r)\left(1+\nu^2\rho^2/2\right)^2+J_2^2(k_f r)\,\nu^4\rho^4/8}{\left(1+\nu^2\rho^2/2\right)^2}, \tag{4}$$

with $\rho^2=\left(p_x^2+p_y^2\right)/p_z^2$. The correlator is plotted in Figure 1 for a number of values of $\nu\rho$.

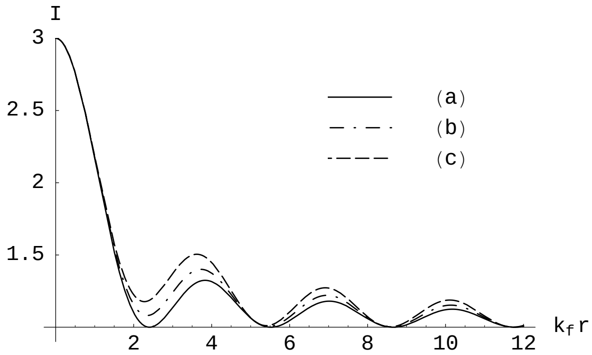

Figure 1: Intensity correlator of a single flexural mode for $\nu\rho$ equal (a) 0, (b) 2, and (c) ∞.

It is equal to 3 at zero separation, $r = 0$, as demanded by Gaussian mode statistics, and is higher than $1 + 2J_0^2(k_f r)$ for non-zero values of $\nu\rho$, with the most pronounced difference observed near the first minimum.

At realistic values of $\nu\rho$ (Schaadt *et al.* estimate $\rho \sim 0.33$; we calculate $\nu = -0.68$ at the relevant frequencies), the effect is small, and difficult to resolve within the data's precision. The sole anomaly in the data is the best-fit value of the relative variance at zero separation: 2.93 ± 0.05, an anomaly with only small statistical significance. Such a value cannot be explained with the current theory; indeed the basic assumption of Gaussian statistics demands that this quantity be 3. However, if the quantity 2.93 is understood as the ratio between the relative variance at zero separation and at the first minimum, then the current theory can explain the anomaly, by calling for $|\nu\rho| = 1.06$.

The data's precision does not support any more detailed comparisons. This is also the case with the in-plane modes. Modes which have even up/down parity consist of an equipartitioned diffuse mixture of longitudinal waves with wavenumber k_l (which have both in-plane and, due to Poisson effect, out-of-plane components of displacement), and in-plane horizontally polarized shear waves with wavenumber k_{sh}. These waves mode convert to one and other at the plate boundaries. Thus the relevant Green's tensor has two wavenumbers, and one anticipates structures like those seen in Figs. 4 and 5 of Schaadt *et al.* However, as in the flexural case, one does not expect to see simple Bessel functions J_0, but rather also terms in J_1 and J_2. The relative amplitudes of these several terms are not obvious *a priori* but could be predicted by the present theory. An attempt to fit their data to the present theory is probably unwarranted at this time. A revisit to structures like theirs, but with a well characterized detector of known polarization, may be indicated.

In summary, we have advanced a modification of Berry Conjecture, appropriate for the eigenmode statistics of wave-bearing systems. It is expected to be relevant, not only for elastic waves in homogeneous plates, but in general statistical physics of waves in heterogeneous and mode-converting systems as well.

Acknowledgment: this work was supported by the National Science Foundation CMS-0201346.

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Appendix A: MULTI-MODE INTENSITY CORRELATOR IN A CHAOTIC PLATE

We start calculation of the full intensity correlator by first considering the normal modes of the Rayleigh-Lamb spectrum [10]. The displacement vector of these modes

is given by

$$\mathbf{u} = \left[U(x_3)\left(k^{-1}\nabla\right) + \hat{\mathbf{x}}_3 W(x_3)\right] f(x_1, x_2),$$

with f satisfying a scalar 2D Helmholtz equation: $\left[\nabla^2 + k^2\right] f(x_1, x_2) = 0$. The displacement components U and W are the solutions of a boundary-value ODE in x_3. With the vertical wavenumbers of longitudinal and shear waves defined as $\alpha^2 = \omega^2/c_l^2 - k^2$, and $\beta^2 = \omega^2/c_t^2 - k^2$, one deduces the dispersion relation for the odd and even up/down parity modes

$$\tan\beta h \,/\, \tan\alpha h = -\left[\left(k^2 - \beta^2\right)^2 / \, 4\alpha\beta k^2\right]^{\pm 1},$$

where $+1$ in the exponent corresponds to the odd parity modes, and -1 to the even modes. The dispersion relation gives the wavenumbers of the odd (k_f) and even modes (k_l) as multi-branched implicit functions of the frequency: $k = k_n(\omega)$. Expressions for U and W of the odd and even modes respectively are

$$\begin{aligned} U =& 2k^3\beta\sin\beta h\sin\alpha x_3 - \left(k^2-\beta^2\right)k\beta\sin\alpha h\sin\beta x_3, \\ W =& 2k^2\alpha\beta\sin\beta h\cos\alpha x_3 + \left(k^2-\beta^2\right)k^2\sin\alpha h\cos\beta x_3; \end{aligned}$$

$$\begin{aligned} U =& 2k^3\beta\cos\beta h\cos\alpha x_3 - \left(k^2-\beta^2\right)k\beta\cos\alpha h\cos\beta x_3, \\ W =& 2k^2\alpha\beta\cos\beta h\sin\alpha x_3 + \left(k^2-\beta^2\right)k^2\cos\alpha h\sin\beta x_3. \end{aligned}$$

By specifying a complete set of solutions f in the plane (for example, standing plane waves or standing cylindrical waves), we construct the modes of an infinite plate. Alternatively, we may specify a complete set of propagating waves f, in which case a complex conjugate must be inserted on the first factor $\mathbf{u}$ in equation (2). These modes are not the natural modes of a finite plate unless the boundary conditions at the outer rim are particularly special. They may nevertheless be used in a modal expansion of the Green's function if attention is confined to early enough times (alternatively if a frequency averaging is done) as discussed above. The average of the exact Green's function, $\mathbf{G}$, can then be substituted by the Green's function in the infinite plate, $\mathbf{G}^\infty$.

We construct a partial Green's function (2) of the Rayleigh-Lamb spectrum, and find its imaginary part,

$$\begin{aligned} \Im G^\infty_{\alpha\beta} =& \sum_n \Big[a_n J_0(k_n r)\,\delta_{\alpha\beta}/2 \\ &+ b_n J_2(k_n r)\left(\delta_{\alpha\beta}/2 - r_\alpha r_\beta/r^2\right)\Big], \\ \Im G^\infty_{33} =& \sum_n c_n J_0(k_n r), \\ \Im G^\infty_{\alpha 3} =& -\Im G_{3\alpha} = \sum_n d_n J_1(k_n r)\, r_\alpha/r. \end{aligned} \tag{A1}$$

The sum is taken over propagating, i.e. having real k_n, modes only. Greek indices span the in-plane space: $\alpha, \beta = \{1, 2\}$. By means of the factor, $\nu_n = U(h)/W(h)|_{k=k_n(\omega)}$, we can write the modal amplitudes, $a_n = b_n = c_n\nu_n^2$, and $d_n = c_n\nu_n$, in terms of the amplitude describing out-of-plane displacement of the plate surface,

$$c_n = \frac{\pi}{4}\frac{\partial k}{\partial\omega}\frac{k}{\omega}\left.\frac{W^2(h)}{\int_{-h}^{+h}\left[U^2(x_3) + W^2(x_3)\right]dx_3}\right|_{k=k_n(\omega)}.$$

The horizontal shear modes have displacements purely in the plane of the plate:

$$\mathbf{u} = V(x_3)\left(k^{-1}\nabla\right)\times\left[\hat{\mathbf{x}}_3 f(x_1, x_2)\right],$$

the dispersion relation for the shear wavenumbers (k_{sh}) being: $k = k_n(\omega) = \sqrt{(\omega/c_t)^2 - (\pi n/2h)^2}$. Imaginary part of the corresponding partial Green's function has the same form as for the Rayleigh-Lamb modes (A1). However, the modal amplitudes are now as follows, $a_n = -b_n = (1+\delta_{0n})/4hc_t^2$, and $c_n = d_n = 0$.

The full multi-mode tensor Green's function includes the modes of all (namely, odd and even parity Rayleigh-Lamb, and horizontal shear) branches required for its short-time expansion at a given frequency. The propagating modes of these branches contribute to the full intensity correlator (3),

$$I(r) = 1 + 2\frac{\left[\sum_n\left(a_n\rho^2/2 + c_n\right)J_0(k_n r)\right]^2 + \left[\sum_n b_n J_2(k_n r)\right]^2\rho^4/8}{\left[\sum_n\left(a_n\rho^2/2 + c_n\right)\right]^2}.$$

The averages over directions of the separation vector $\mathbf{r}$ are carried out with the help of the following rules:

$$\begin{aligned} \left\langle r_\alpha r_\beta/r^2\right\rangle =& \delta_{\alpha\beta}/2, \\ \left\langle r_\alpha r_\beta r_\gamma r_\iota/r^4\right\rangle =& \left(\delta_{\alpha\beta}\delta_{\gamma\iota} + \delta_{\alpha\gamma}\delta_{\beta\iota} + \delta_{\alpha\iota}\delta_{\beta\gamma}\right)/8 \end{aligned}$$

In the special case that we have, the frequency is such that there is only one odd (flexural) mode, and the sum is replaced with a single term, yielding correlator (4). The factor ν is computed for the plate parameters of Ref. [7] (thickness 3 mm, Poisson ratio 0.33, transverse wavespeed, $c_t = 3.1\,\mathrm{mm}/\mu\mathrm{s}$), and excitation frequencies 432, 510, 513, 514 kHz, to be $\nu = -0.68$.